

# On Semi-Periods

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## ABSTRACT

The periods of the three-form on a Calabi–Yau manifold are found as solutions of the Picard–Fuchs equations; however, the toric varietal method leads to a generalized hypergeometric system of equations, first introduced by Gelfand, Kapranov and Zelevinski, which has more solutions than just the periods. This same extended set of equations can be derived from symmetry considerations. Semi-periods are solutions of the extended GKZ system. They are obtained by integration of the three-form over chains; these chains can be used to construct cycles which, when integrated over, give periods. In simple examples we are able to obtain the complete set of solutions for the GKZ system. We also conjecture that a certain modification of the method will generate the full space of solutions in general.

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## 1. Preamble

The moduli space of a Calabi–Yau manifold naturally splits into two spaces: the space  $M_{21}$  of complex structure parameters, and the space  $M_{11}$  of (complexified) Kähler class parameters. Calculations involving the complex structure parameters are exact, while those involving the Kähler class parameters are corrected by instantons (for a comprehensive review, see [1]). Nevertheless, under mirror symmetry,  $M_{11}$  of one manifold is related to  $M_{21}$  of the mirror manifold. Thus the parameters of the low energy effective action of a string theory compactified on a Calabi–Yau manifold can be calculated exactly by studying the space  $M_{21}$  of the manifold, and the space  $M_{21}$  of its mirror.

Since the discovery that Special Geometry applies to the moduli space of Calabi–Yau manifolds [2,3,4], the calculation of the periods of the three-form has become something of an industry. The space of complex structures is fully described by the set of periods, in the sense that by knowing the periods, one can calculate the metric on the moduli space and hence the kinetic term and the Yukawa couplings in the low energy effective action.

Begin with the case of a family of hypersurfaces  $\mathcal{M}_\phi$  defined as the zero-set of a defining polynomial  $p_\phi$ . The periods are defined as

$$\varpi_j(\phi_\alpha) \stackrel{\text{def}}{=} \int_{\gamma^j} \Omega(\phi_\alpha) , \quad (1.1)$$

where  $\Omega(\phi_\alpha)$  is the nowhere vanishing holomorphic three-form [5,6]

$$\Omega(\phi_\alpha) \stackrel{\text{def}}{=} \text{Res}_{\mathcal{M}_\phi} \left[ \frac{(x d^4 x)}{p_\phi} \right]$$

on the Calabi-Yau three-fold specified by the parameters  $\phi_\alpha$ , and  $\gamma^j$  is a three-cycle labelled by  $j$ . The four-differential  $(x d^4 x)$  is the ‘natural’ one: on a weighted projective  $N$ -space  $\mathbb{P}_{(k_1, \dots, k_{N+1})}^N$ , we have

$$(x d^N x) \stackrel{\text{def}}{=} \frac{1}{(N+1)!} \epsilon^{i_1 i_2 \dots i_{N+1}} k_{i_1} x_{i_1} dx_{i_2} \dots dx_{i_{N+1}} ,$$

where  $k_i$  are the weights of the coordinates. That is, in  $\mathbb{P}_{(k_1, \dots, k_{N+1})}^N$ ,

$$(x_1, \dots, x_{N+1}) \cong (\lambda^{k_1} x_1, \dots, \lambda^{k_{N+1}} x_{N+1}) , \quad \lambda \in \mathbb{C}^* .$$

The period (1.1) will here be calculated by choosing one of the standard coordinate patches in  $\mathbb{P}^4_{(k_1, \dots, k_5)}$ ,  $\mathcal{U}_m$ , where  $x_m \neq 0$  so that  $x_m = 1$  by projectivity. There [7]:

$$\varpi_j^{(m)}(\phi_\alpha) = C \int_{\Gamma^j} \frac{\prod_{i \neq m} dx_i}{p_\phi|_m}, \quad (1.2)$$

with  $C$  a convenient prefactor. This is easily seen to apply for Calabi-Yau weighted hypersurfaces of arbitrary dimension.

We separate the polynomial into a reference polynomial  $p_0$  independent of the moduli  $\phi$ , and a perturbative part  $\Delta$  which does depend on the  $\phi$ . This amounts to choosing a reference point in moduli space, and expanding around that point. The basic idea here is, while working in the patch  $\mathcal{U}_m$  but suppressing the label, to expand  $1/p_\phi$  around  $1/p_0$ , and utilize the Laplace transform

$$\frac{1}{p_\phi} = \frac{1}{p_0 + \Delta} = \sum_{n=0}^{\infty} \frac{(-\Delta)^n}{(p_0)^{n+1}} = \frac{d}{k_m} \int_0^\infty dx_m \sum_n \frac{(-\Delta)^n}{n!} e^{-p_0 x_m}, \quad \Re(p_0) > 0, \quad (1.3)$$

which produces a “small- $\phi$ ” expansion of the periods (1.2). We have relabelled the Laplace transform parameter to  $x_m^{d/k_m}$ , and so the expansion appears as if in homogeneous coordinates. With a choice of the poly-contours  $\{\Gamma^j\}$ , Eq. (1.2) may be considered a definition of the periods.

The semi-period is a building block for a period. In other words, it is the integral of the three-form over a chain instead of a cycle, with the understanding that the chain can be manipulated in some way to build cycles. The semi-period construction we will discuss is naturally related to the period construction in [7]. The prototypical semi-period we write as

$$F_0 = \int_V d^{N+1}x \sum_{n=0}^{\infty} \frac{(-\Delta)^n}{n!} e^{-p_0 x}, \quad (1.4)$$

where  $V$  is the  $N+1$ -chain  $\{x_1, \dots, x_{N+1} \text{ real and positive}\}$ .

As was worked out in [7], these chains are enough to build cycles and thus to calculate periods. The advantage of this approach is its ease of calculation of a full set of periods in different regions of moduli space. This method lends itself to many different constructions: all the types of polynomial hypersurfaces discussed in [8,9], (weighted) complete intersection spaces [1, 10, 11, 12, 13], generalized Calabi-Yau manifolds [14, 15], Landau-Ginzburg vacua [16,17,18], and toric varietal constructions related to these and other types [19,20,21,22].

A semi-period is also the solution of a generalized hypergeometric system of equations. The periods are solutions of the Picard-Fuchs (PF) equations. In the toric varietal

approach, differential equations are constructed based on the points in the dual polyhedron and the generators of Mori cone [19,20,23]. The system of equations obtained in this manner has extra solutions beside periods, since the complete set of solutions is larger than the set of linearly independent periods. We show by construction that semi-periods such as (1.4) are such extra solutions. All of the periods can be obtained either as the solutions to PF equations, or as linear combinations of semi-periods.

In many if not all cases the same generalized hypergeometric system that was obtained from toric data can be constructed based on the symmetries of the period, where the period is expressed as an integral over a cycle [20,24]. These equations are also satisfied by a semi-period, which is expressed using the same integrand, but integrated over a particular chain, not a cycle. Not every chain yields a solution to the differential equations; for those that do, we call the integral a semi-period.

Our aim in this paper is to construct, calculate and investigate the semi-period solutions to the  $\Delta^*$  hypergeometric system of equations. The layout of the paper is as follows: Section 2 describes the construction of the generalized hypergeometric system of equations using the toric varietal approach and Section 3 illustrates the procedure by doing a simple example: the  $\mathbb{Z}_3$  torus. An interesting set of examples are explored in Sections 4 and 5.

## 2. Construction of the $\Delta^*$ Hypergeometric System associated to Toric Manifolds

In this section we briefly review the construction of the  $\Delta^*$  hypergeometric system for Calabi–Yau manifolds described as the zero loci of homogeneous polynomials in weighted projective spaces.

We denote a weighted projective space as  $\mathbb{P}_{\mathbf{k}}^r$ , by which we mean the set of complex  $x_i$  (not all zero) identified under  $(x_1, x_2, \dots, x_{r+1}) \sim (\lambda^{k_1} x_1, \lambda^{k_2} x_2, \dots, \lambda^{k_{r+1}} x_{r+1})$  for all  $\lambda \neq 0$ . Consider the Calabi–Yau manifold  $\mathbb{P}_{\mathbf{k}}^r[d]$  with  $\sum_{i=1}^{r+1} k_i = d$ , where  $d$  is the degree of the defining polynomial whose general expression is :

$$p = \sum_{\mathbf{m}} c_{\mathbf{m}} x^{\mathbf{m}}$$

Here  $\mathbf{m}$  is the degree vector such that  $\mathbf{m} \cdot \mathbf{k} = d$ .

We associate to each monomial a point with coordinates  $\mathbf{m} = (m_1, \dots, m_{r+1})$  in the lattice  $\mathbb{Z}^{r+1}$ . The set of all monomials compatible with  $\mathbf{m} \cdot \mathbf{k} = d$  will define a convex polyhedron  $\Delta$  in  $\mathbb{Z}^{r+1} \otimes \mathbb{R}$ , which is moreover reflexive (in the sense of [19]). This polyhedron lies in a hyperplane defined by the weight vector normal to it and will contain all the points of the sublattice that lie within its boundaries. By uniting the vertices of

the polyhedron  $\Delta$  with the origin we obtain a cone whose dual can be easily constructed by the method described in [19].

The vertices of the dual polyhedron  $\Delta^*$  are given by the first intersection point between the generators of the dual cone and the dual lattice. Because none of the weights in the definition of  $\mathbb{P}_{\mathbf{k}}^r$  is zero, we can project the dual polyhedron on a  $\mathbb{Z}^r \otimes \mathbb{R}$  subspace and then find a basis for the sublattice defined by the projected points of the dual polyhedron. We take the unique interior point as the origin of the new coordinate system, and then put back the dual polyhedron in  $\mathbb{Z}^{r+1} \otimes \mathbb{R}$  along the hyperplane  $x_0 = 1$ .

This leaves us with the following set of dual points:

$$\bar{\nu}_i^* = (1, \nu_i^{*1}, \dots, \nu_i^{*r}) \quad i = 1, \dots, p \quad , \quad (2.1)$$

and the interior point

$$\bar{\nu}_0^* = (1, 0, \dots, 0) \quad .$$

The space of linear dependence relations between the dual points,  $\sum_{i=0}^p l_i^a \bar{\nu}_i^* = 0$ , will have dimension  $p - r$ . Nevertheless as Batyrev remarked in [19] we are interested only in points that do not lie in the interior of codimension 1 faces. Assuming there are  $n + 1$  of them, we denote  $A = \{\bar{\nu}_0^*, \bar{\nu}_{i_1}^*, \dots, \bar{\nu}_{i_n}^*\}$  the subset of such points in the dual polyhedron. The linear dependences between points in  $A$  are described by the lattice of rank  $n - r$ :

$$L = \{(l_0, \dots, l_n) \in \mathbb{Z}^{n+1} \mid \sum_{i=0}^n l_i^a \bar{\nu}_i^* = 0, \bar{\nu}_i^* \in A\} \quad (2.2)$$

Considering the affine complex space  $\mathbb{C}^{p+1}$  with coordinates  $(a_0, \dots, a_p)$  we define a consistent system of differential operators:

$$\begin{aligned} Z_j &= \sum_{i=0}^n \bar{\nu}_{i,j}^* a_i \frac{\partial}{\partial a_i} - \beta_j \\ D_{1^a} &= \prod_{l_i > 0} \left( \frac{\partial}{\partial a_i} \right)^{l_i^a} - \prod_{l_i < 0} \left( \frac{\partial}{\partial a_i} \right)^{-l_i^a} ; \quad 1^a \in L \end{aligned} \quad (2.3)$$

The exponent of  $\beta$  that appears in the definition of the differential operators  $Z_j$  will be taken to be  $(-1, 0, \dots, 0)$ . With this choice the period integrals vanish when acted upon by the operators (2.3) [19].

The set of operators (2.3) taken together form the  $\Delta^*$  hypergeometric system of equations. This is an example of a class of generalized hypergeometric equations described by Gel'fand, Kapranov and Zelevinski [25], and hence is also called a GKZ system. In general,

the  $\Delta^*$  system is not enough to determine the Picard-Fuchs equations; it must be extended by supplementing further differential operators. The methods for doing this are discussed in [20, 23, 26]. In other words, there are more solutions to this system of equations than just the periods. We conjecture that semi-periods form a complete set of solutions.

The periods and semi-periods depend on  $a_i$  through  $\zeta^a$  [20], defined as:

$$\zeta^a = (-1)^{\hat{l}_0^a} \mathbf{a}^{\hat{\mathbf{l}}^a} . \quad (2.4)$$

We used  $\hat{\mathbf{l}}^a$  to denote the generators of the Mori cone which can be found by standard methods(see for example [23]). The variables  $\zeta^a$  are not only the natural choices in order to satisfy the linear constraints imposed by the  $Z^j$ , but are also good coordinates on the moduli space for describing the large complex structure limit of the mirror manifold.

### 3. Simple Example

#### 3. 1. The $\Delta^*$ Hypergeometric System for the $\mathbb{Z}_3$ Torus

The  $\mathbb{Z}_3$  torus can be described as a cubic hypersurface in  $\mathbb{P}^2$ . The vector of weights is  $\mathbf{k} = (1, 1, 1)$ , and so the vertices of the dual polyhedron become

$$\bar{\nu}_1^* = (1, -1, -1) \quad \bar{\nu}_2^* = (1, 1, 0) \quad \bar{\nu}_3^* = (1, 0, 1) ,$$

and the interior point is  $\bar{\nu}_0^* = (1, 0, 0)$ .

There is one relation of the form  $\sum_i l_i \bar{\nu}_i^* = 0$ , satisfied by  $\mathbf{l} = (-3, 1, 1, 1)$ . Taking this into the account, the differential operators defined by (2.3) will be of the following form:

$$D = \frac{\partial}{\partial a_1} \frac{\partial}{\partial a_2} \frac{\partial}{\partial a_3} - \left( \frac{\partial}{\partial a_0} \right)^3 \quad (3.1)$$

and

$$\begin{aligned} Z_1 &= \left( \sum_{i=0}^3 a_i \frac{\partial}{\partial a_i} + 1 \right) \\ Z_2 &= \left( a_2 \frac{\partial}{\partial a_2} - a_1 \frac{\partial}{\partial a_1} \right) \\ Z_3 &= \left( a_3 \frac{\partial}{\partial a_3} - a_1 \frac{\partial}{\partial a_1} \right) . \end{aligned} \quad (3.2)$$

The equations  $Z_i \Pi = 0$  are solved by  $\Pi$  of the form

$$\Pi = \frac{1}{a_0} \tilde{\Pi} \left( -\frac{a_1 a_2 a_3}{a_0^3} \right) ,$$

and the equation  $D\Pi = 0$  becomes

$$[\Theta^3 - \zeta(3\Theta + 3)(3\Theta + 2)(3\Theta + 1)] \tilde{\Pi} = 0 , \quad (3.3)$$

where  $\Theta = \zeta \frac{\partial}{\partial \zeta}$ , and  $\zeta$  is the argument of  $\tilde{\Pi}$ , found from (2.4).

Solving the generalized hypergeometric equation one has [27] :

$$\tilde{\Pi} = \mathcal{P} \left\{ \begin{matrix} 0 & \infty & 1 \\ 0 & 1/3 & 0 \\ 0 & 2/3 & 1 & 3^3 \zeta \\ 0 & 1 & 0 \end{matrix} \right\} .$$

We will be interested in the region where  $a_0$  is small. Since  $\zeta$  is related to  $a_0^{-3}$ , we should look at the solutions near  $\zeta \rightarrow \infty$ . Remembering that  $\Pi = \frac{1}{a_0} \tilde{\Pi}$ , and setting  $a_1 = a_2 = a_3 = 1$ ,  $\zeta = -1/a_0^3$ , the solutions to the  $\Delta^*$ -hypergeometric system are

$$\begin{aligned} \Pi_0 &= {}_2F_1 \left( \frac{1}{3}, \frac{1}{3}; \frac{2}{3}; -\left(\frac{a_0}{3}\right)^3 \right) \\ \Pi_1 &= a_0 {}_2F_1 \left( \frac{2}{3}, \frac{2}{3}; \frac{4}{3}; -\left(\frac{a_0}{3}\right)^3 \right) \\ \Pi_2 &= a_0^2 {}_3F_2 \left( 1, 1, 1; \frac{4}{3}, \frac{5}{3}; -\left(\frac{a_0}{3}\right)^3 \right) . \end{aligned} \quad (3.4)$$

### 3. 2. The Equation from Symmetry Considerations

Suppose we look at the period defined as

$$\Pi = \int_{\Gamma} \frac{x_1 dx_2 dx_3}{a_1 x_1^3 + a_2 x_2^3 + a_3 x_3^3 + a_0 x_1 x_2 x_3} , \quad (3.5)$$

where  $\Gamma$  is some cycle. Since  $\Pi$  is independent of  $x_1$ , it is equivalent to

$$\Pi = \frac{1}{2\pi i} \int_{\Gamma'} \frac{dx_1 dx_2 dx_3}{a_1 x_1^3 + a_2 x_2^3 + a_3 x_3^3 + a_0 x_1 x_2 x_3} , \quad (3.6)$$

where  $\Gamma' = C_1 \times \Gamma$ , and  $C_1$  is a loop around  $x_1 = 0$ . Now look at the symmetries of  $\Pi$  as a function of the  $a_i$ :

$$\Pi(a_0, \lambda^3 a_1, \lambda^{-3} a_2, a_3) = \Pi(a_0, a_1, a_2, a_3) , \quad (3.7)$$

and the same with  $a_2 \rightarrow a_3$ , so

$$\begin{aligned} a_1 \frac{\partial \Pi}{\partial a_1} &= a_2 \frac{\partial \Pi}{\partial a_2} \\ a_1 \frac{\partial \Pi}{\partial a_1} &= a_3 \frac{\partial \Pi}{\partial a_3} , \end{aligned} \tag{3.8}$$

while

$$\Pi(\lambda^3 a_0, \lambda^3 a_1, \lambda^3 a_2, \lambda^3 a_3) = \lambda^{-3} \Pi(a_0, a_1, a_2, a_3) \tag{3.9}$$

implies that

$$a_0 \frac{\partial \Pi}{\partial a_0} + \sum_{i=1}^3 a_i \frac{\partial \Pi}{\partial a_i} = -\Pi . \tag{3.10}$$

These are the  $Z_i$  equations (3.2). Finally, since  $x_1^3 x_2^3 x_3^3 = (x_1 x_2 x_3)^3$ , the period satisfies the relation

$$\frac{\partial}{\partial a_1} \frac{\partial}{\partial a_2} \frac{\partial}{\partial a_3} \Pi = \left( \frac{\partial}{\partial a_0} \right)^3 \Pi , \tag{3.11}$$

which is the  $D$  equation (3.3).

Thus we are able to find the equations from consideration of the symmetries of the period, without going through the toric variety construction.

### 3. 3. The Semi-Periods for the $\mathbb{Z}_3$ Torus

Consider the integrand in the definition of the period (3.5), and note that we can take a Laplace transform as long as  $\Re e(p) > 0$ :

$$\frac{x_1 dx_2 dx_3}{p} = \int_{x_1=0}^{\infty} d^3 x e^{-p} .$$

Taking the polynomial to be

$$p = a_1 x_1^3 + a_2 x_2^3 + a_3 x_3^3 + a_0 x_1 x_2 x_3 ,$$

we define the integration contour  $V = \{x_1, x_2, x_3 \text{ real and positive}\}$  and construct

$$F = \int_V d^3 x e^{-p} . \tag{3.12}$$

$F$  as defined also satisfies the relations (3.8) (3.10) (3.11) derived from symmetry considerations above, and hence is the extra solution besides the periods. This is a semi-period. By using the symmetries (3.7) (3.9), we bring the polynomial in the integrand of  $F$  to canonical form

$$p = x_1^3 + x_2^3 + x_3^3 + \phi x_1 x_2 x_3 .$$



Now it is easy to calculate  $F(\phi)$ :

$$\begin{aligned} F &= \sum_{n=0}^{\infty} \frac{(-\phi)^n}{n!} \prod_{i=1}^3 \int_0^{\infty} dx_i x_i^n e^{-x_i^3} \\ &= \frac{1}{3^2} \sum_{n=0}^{\infty} \frac{(-\phi)^n}{n!} \Gamma^3\left(\frac{n+1}{3}\right) \end{aligned} \quad (3.13)$$

which is, of course, a linear combination of  $\Pi_0$ ,  $\Pi_1$  and  $\Pi_2$  of (3.4), with  $a_0 = \phi$ .

But now, consider integrating over a slightly different contour. Let  $x_2$  run from 0 to  $\omega\infty$ ,  $\omega$  being a cube root of unity, and call this new contour  $A_2V$ . This still satisfies the equations, but by a change of variables we see that

$$\int_{A_2V} d^3x e^{-p(\phi)} = \omega F(\omega\phi) .$$

There are three semi-periods we can form in this way, summarized by integrating  $x_2$  from 0 to  $\omega^\delta\infty$ ,  $\delta = 0, 1, 2$ . This inserts a factor of  $\omega^{n+1}$  in the summation of (3.13), and leads to three independent linear combinations of (3.4).

So what happens when we form a spoke in the  $x_2$  plane, that is, we integrate  $x_2$  along a contour that comes in infinity along one of these paths, and returns along the other? The integration contour is  $V - A_2V$ , so the result of the integration is  $F(\phi) - \omega F(\omega\phi)$ , which is a linear combination of  $\Pi_0$  and  $\Pi_1$ . Note that in generating periods for the torus by using spokes [7], we take exactly these sorts of contours, and the periods are linear combinations of  $\Pi_0$  and  $\Pi_1$ . Details of this example are in the Appendix in [14]. A cycle could be generated, for example, by integrating over  $(1 - A_2)(1 - A_3)V$ , and the period that results is  $F(\phi) - 2\omega F(\omega\phi) + \omega^2 F(\omega^2\phi)$ .

Why do the semi-periods satisfy the  $\Delta^*$  hypergeometric system? The differential equations can be generated by considering the symmetries of the period integral (3.6). The symmetry in (3.7) is demonstrated by a change of variables in the integrand of (3.6),  $x_1 \rightarrow \lambda x_1$  and  $x_2 \rightarrow \lambda^{-1}x_2$ . Since the region of integration has no boundary, the expression is the same. Applying the same consideration to (3.12) or (3.13), we see that (taking  $\lambda$  to be real), the boundary is unchanged and again, the expression is the same. Thus the same symmetries are exhibited by the semi-periods, since the boundary of the chain has been chosen properly.

The  $\Delta^*$  system contains the Picard-Fuchs equations, and is thus solved by the periods; but it is also solved by the semi-periods. The methods for reducing the order of the  $\Delta^*$  system to the order of the PF equations are discussed in [20,23,26]. In [23], use is made of further properties of the periods that can be derived using partial integration. These properties are not shared by the semi-periods, and hence this method naturally selects

periods over semi-periods. References [20] and [26] find other ways of selecting periods, either by reducing the order of the  $\Delta^*$  system, or imposing additional differential equations.

There are, of course, more semi-periods than periods. Even though we have not been able to explicitly construct them in general, we believe that all solutions of the  $\Delta^*$  system can be obtained by integrating the three-form over well chosen chains. We have not found a general principle that guarantees that all solutions to the  $\Delta^*$  system can be found as integrals over chains — it is, however, true in simple cases. It seems likely that the selection of these chains is highly model dependent. For the types of chains we consider, the number of semi-periods generated in a particular corner of moduli space depends on the symmetries of  $p_0$ , the reference polynomial.

#### 4. A Known Example

Here is an example for which the periods are discussed in several places, namely Section 4.2 of [28], Section 3.1 of [7] and Section 3.1 of [23]. Let us examine the semi-periods for a particular family of hypersurfaces,  $\mathbb{P}_{(2,2,3,7,7)}[21]^{50,11}$ , where the superscripts indicate the Hodge numbers  $b_{2,1} = 50$  and  $b_{1,1} = 11$ , with the polynomial chosen as

$$p = a_1 y_1^7 y_4 + a_2 y_2^7 y_5 + a_3 y_3^7 + a_4 y_4^3 + a_5 y_5^3 + a_0 y_1 y_2 y_3 y_4 y_5 + a_6 (y_1 y_2 y_3)^3 .$$

To calculate a semi-period  $F$ , we must pick a corner of moduli space. Supposing we treat the last two terms in  $p$  as perturbations,  $F$  is calculated to be

$$\begin{aligned} F &= \sum_{m,k=0}^{\infty} \frac{(-a_0)^m}{m!} \frac{(-a_6)^k}{k!} \\ &\quad \times \int_0^{\infty} d^5 y (y_1 y_2 y_3)^{m+3k} (y_4 y_5)^m e^{-(a_1 y_1^7 y_4 + a_2 y_2^7 y_5 + a_3 y_3^7 + a_4 y_4^3 + a_5 y_5^3)} \quad (4.1) \\ &= \frac{1}{3^2 7^3} \sum_{\substack{m,k=0 \\ 2m+2>k}}^{\infty} \frac{(-a_0)^m}{m!} \frac{(-a_6)^k}{k!} \frac{\Gamma^3\left(\frac{m+3k+1}{7}\right)}{(a_1 a_2 a_3)^{\frac{m+3k+1}{7}}} \frac{\Gamma^2\left(\frac{2m-k+2}{7}\right)}{(a_4 a_5)^{\frac{2m-k+2}{7}}} . \end{aligned}$$

Things to notice about this in reference to the references [28,7,23]:

- This can be expressed as a function of the two large complex structure parameters as defined in [28,23],  $\zeta_1 = -\frac{a_4 a_5 a_6}{a_0^3}$  and  $\zeta_2 = -\frac{a_1 a_2 a_3}{a_0 a_6^2}$ , via

$$\begin{aligned} F &= \frac{1}{a_0} \tilde{F}(\zeta_1, \zeta_2) \\ \tilde{F} &= \sum_{r=0}^1 \sum_{j,l=0}^{\infty} \left(-\frac{1}{\zeta_1}\right)^{\frac{2j+2-r}{7}} \left(-\frac{1}{\zeta_2}\right)^{l+\frac{j+1+3r}{7}} \frac{(-1)^{l+j+r}}{\Gamma(l+j+1)\Gamma(2l+1+r)} \quad (4.2) \\ &\quad \times \Gamma^3\left(l+\frac{j+1+3r}{7}\right) \Gamma^2\left(\frac{2j+2-r}{7}\right) . \end{aligned}$$

$F$  has been calculated in an expansion around large  $\zeta_1$  and  $\zeta_2$ .

- This obeys the equations found in [23],

$$\begin{aligned} \partial_{a_0}^3 F - \partial_{a_4} \partial_{a_5} \partial_{a_6} F &= 0 \\ \partial_{a_6}^2 \partial_{a_0} F - \partial_{a_1} \partial_{a_2} \partial_{a_3} F &= 0 , \end{aligned} \tag{4.3}$$

which become the extended Picard–Fuchs equations

$$\begin{aligned} \left\{ \Theta_1^2 (\Theta_1 - 2\Theta_2) - \zeta_1 (3\Theta_1 + \Theta_2 + 3) (3\Theta_1 + \Theta_2 + 2) (3\Theta_1 + \Theta_2 + 1) \right\} \tilde{F} &= 0 \\ \left\{ \Theta_2^3 - \zeta_2 (\Theta_1 - 2\Theta_2 - 1) (\Theta_1 - 2\Theta_2) (3\Theta_1 + \Theta_2 + 1) \right\} \tilde{F} &= 0 , \end{aligned}$$

using  $\Theta_i = \zeta_i \partial_{\zeta_i}$ .

- There are seven semi-periods that are easy to find in this expansion. These were exploited in [7] to calculate the periods. The different semi-periods are obtained by integrating over different chains in (4.1), as detailed in [7]. Consider changing the region of integration in (4.1) so that  $y_1$  runs from 0 to  $\lambda\infty$ , where  $\lambda$  is a seventh root of unity. The net effect is to change  $F$  by

$$A : F(a_0, a_6) \rightarrow \lambda F(\lambda a_0, \lambda^3 a_6) .$$

Explicitly, this leads to the insertion of  $\lambda^{m+3k+1}$  inside the summation in (4.1), or an insertion of  $\lambda^{j+1+3r}$  inside the summation in (4.2). This leads to seven semi-periods. It has been shown in [7] that the six independent periods can be calculated using spokes, that is, using differences of semi-periods generated in this way. The action of  $A$  is discussed in [28,7].

## 5. Nesting Example

The following example is interesting because it shows how  $K3$  and  $T^2$  can appear. In this section we extend an argument by Klemm, Lerche and Mayr [29] to include the semi-periods; see also [30] and [20]. Consider the manifold  $\mathbb{P}_{(1,1,2,4,4)}[12]^{101,5}$ , and choose a particular family of hypersurfaces,

$$p = a_1 x_1^{12} + a_2 x_2^{12} + a_3 x_3^6 + a_4 x_4^3 + a_5 x_5^3 + a_0 x_1 x_2 x_3 x_4 x_5 + a_6 (x_1 x_2)^6 + a_7 (x_1 x_2 x_3)^3 .$$

This is a  $K3$  fibration, as can be seen by making the substitutions  $x_2 = \lambda x_1$ ,  $\tilde{x}_1 = x_1^2$ , which leads to

$$p = \tilde{x}_1^6 (a_1 + \lambda^{12} a_2 + \lambda^6 a_6) + a_3 x_3^6 + a_4 x_4^3 + a_5 x_5^3 + \lambda a_0 \tilde{x}_1 x_3 x_4 x_5 + \lambda^3 a_7 (\tilde{x}_1 x_3)^3 .$$

The hypersurface  $p = 0$  describes a  $K3$  hypersurface  $\mathbb{P}_{(1,1,2,2)}$ [6]. By a similar substitution, this  $K3$  is shown to be an elliptic fibration over  $\mathbb{P}^1$  with generic fiber  $\mathbb{P}_{(1,1,1)}$ [3], otherwise known as the  $\mathbb{Z}_3$  torus, described in Section 3.

The dual polyhedron of  $\mathbb{P}_{(1,1,2,4,4)}$ [12] contains 8 points:

$$\begin{aligned}\bar{\nu}_0^* &= (1, 0, 0, 0, 0) \\ \bar{\nu}_1^* &= (1, 4, 0, 2, 3) \\ \bar{\nu}_2^* &= (1, 0, 0, 0, 1) \\ \bar{\nu}_3^* &= (1, 0, 0, 1, 0) \\ \bar{\nu}_4^* &= (1, 0, 1, 0, 0) \\ \bar{\nu}_5^* &= (1, -1, -1, -1, -1) \\ \bar{\nu}_6^* &= (1, 2, 0, 1, 2) \\ \bar{\nu}_7^* &= (1, 1, 0, 1, 1)\end{aligned}$$

We find 6 vectors in the Mori cone, which can all be expressed with positive integer coefficients in terms of the following basis:

$$\begin{aligned}\mathbf{l}^1 &= (-3, 0, 0, 0, 1, 1, 0, 1) \\ \mathbf{l}^2 &= (0, 0, 0, 1, 0, 0, 1, -2) \\ \mathbf{l}^3 &= (0, 1, 1, 0, 0, 0, -2, 0)\end{aligned}$$

The large complex structure coordinates are:

$$\begin{aligned}\zeta_1 &= -\frac{a_4 a_5 a_7}{a_0^3} \\ \zeta_2 &= \frac{a_3 a_6}{a_7^2} \\ \zeta_3 &= \frac{a_1 a_2}{a_6^2}\end{aligned}$$

In this particular example the constraint operators  $Z_j$  have the expressions:

$$\begin{aligned}Z_1 &= \sum_{i=0}^7 a_i \frac{\partial}{\partial a_i} + 1 \\ Z_2 &= 4a_1 \frac{\partial}{\partial a_1} - a_5 \frac{\partial}{\partial a_5} + 2a_6 \frac{\partial}{\partial a_6} + a_7 \frac{\partial}{\partial a_7} \\ Z_3 &= a_4 \frac{\partial}{\partial a_4} - a_5 \frac{\partial}{\partial a_5} \\ Z_4 &= 2a_1 \frac{\partial}{\partial a_1} + a_3 \frac{\partial}{\partial a_3} - a_5 \frac{\partial}{\partial a_5} + a_6 \frac{\partial}{\partial a_6} + a_7 \frac{\partial}{\partial a_7} \\ Z_5 &= 3a_1 \frac{\partial}{\partial a_1} + a_2 \frac{\partial}{\partial a_2} - a_5 \frac{\partial}{\partial a_5} + 2a_6 \frac{\partial}{\partial a_6} + a_7 \frac{\partial}{\partial a_7}\end{aligned}$$

The  $D_1$  operators are:

$$\begin{aligned} D_1 &= \frac{\partial}{\partial a_1} \frac{\partial}{\partial a_2} - \left( \frac{\partial}{\partial a_6} \right)^2 \\ D_2 &= \frac{\partial}{\partial a_4} \frac{\partial}{\partial a_5} \frac{\partial}{\partial a_7} - \left( \frac{\partial}{\partial a_0} \right)^3 \\ D_3 &= \frac{\partial}{\partial a_3} \frac{\partial}{\partial a_6} - \left( \frac{\partial}{\partial a_7} \right)^2 \end{aligned}$$

If we parametrize the solutions of the above system as follows:

$$F(\mathbf{a}) = \frac{1}{a_0} \tilde{F}(\zeta_i)$$

then  $\tilde{F}(\zeta_i)$  is a solution of  $D_i \tilde{F} = 0$ , with

$$\begin{aligned} \mathcal{D}_1 &= \theta_3^2 - \zeta_3 \prod_{i=0}^1 (\theta_2 - 2\theta_3 - i) \\ \mathcal{D}_2 &= \theta_1^2 (\theta_1 - 2\theta_2) - 3^3 \zeta_1 \prod_{i=1}^3 (\theta_1 + i/3) \\ \mathcal{D}_3 &= \theta_2 (\theta_2 - 2\theta_3) - \zeta_2 \prod_{i=0}^1 (\theta_1 - 2\theta_2 - i) \end{aligned} \tag{5.1}$$

where  $\theta_i = \zeta_i \frac{\partial}{\partial \zeta_i}$ .

The dimension of the space of solutions for the  $\Delta^*$ -generalized hypergeometric system is finite and can be calculated by two methods.

The first method uses the associated indicial equations. In the corner of the moduli space where all the  $\zeta_i$ 's are large, we obtain the following system of indicial equations (after the change of variables  $\zeta_i = \frac{1}{\zeta_i}$ ):

$$\begin{aligned} \prod_{i=0}^1 (-\tilde{s}_2 + 2\tilde{s}_3 - i) &= 0 \\ \prod_{i=1}^3 (-\tilde{s}_1 + i/3) &= 0 \\ \prod_{i=0}^1 (-\tilde{s}_1 + 2\tilde{s}_2 - i) &= 0 \quad . \end{aligned} \tag{5.2}$$

The above system has 12 solutions. It was obtained by acting with the system of operators (5.1) on  $\tilde{F}(\tilde{\zeta}_i)$  given in the following form:

$$\tilde{F}(\tilde{\zeta}_i) = \tilde{\zeta}_1^{\tilde{s}_1} \tilde{\zeta}_2^{\tilde{s}_2} \tilde{\zeta}_3^{\tilde{s}_3} \sum_{a,b,c=0}^{\infty} C_{abc} \tilde{\zeta}_1^a \tilde{\zeta}_2^b \tilde{\zeta}_3^c \quad \text{for } \tilde{\zeta}_i \rightarrow 0 \quad . \tag{5.3}$$

The other method of counting the number of solutions is by calculating the volume of the dual cone [25]. When the dual polyhedron is a simplex, the volume is the absolute value of the determinant of the generators of the dual cone (the volume of the  $n$ -dimensional cube is taken to be  $n!$ ). Otherwise, we can use the simplicial decomposition used to find the generators of the Mori cone, and then add up the volumes of the different simplices.

Taking the last two differential operators from (5.1), but dropping the  $\Theta_3$  term in the second, we are left with the  $\Delta^*$  hypergeometric system for the  $K3$  hypersurface  $\mathbb{P}_{(1,1,2,2)}[6]$ . To see this we have to do the same type of analysis for  $\mathbb{P}_{(1,1,2,2)}[6]$ . The dual polyhedron contains 6 points:

$$\begin{aligned}\bar{\nu}_0^* &= (1, 0, 0, 0) \\ \bar{\nu}_1^* &= (1, -1, -1, -1) \\ \bar{\nu}_2^* &= (1, 1, -1, -1) \\ \bar{\nu}_3^* &= (1, 0, 1, 0) \\ \bar{\nu}_4^* &= (1, 0, 0, 1) \\ \bar{\nu}_5^* &= (1, 0, -1, -1)\end{aligned}$$

out of which one is interior to a 1-face, so we are going to study the family:

$$p = \sum_{i=1}^4 a_i x_i^{\frac{6}{k_i}} + a_0 \prod_{i=1}^4 x_i + a_5 x_1^3 x_2^3$$

The Mori cone has two generators:

$$\begin{aligned}\mathbf{l}^1 &= (-3, 0, 0, 1, 1, 1) \\ \mathbf{l}^2 &= (0, 1, 1, 0, 0, -2)\end{aligned}$$

The large complex structure parameters are:

$$\begin{aligned}\zeta_1 &= -\frac{a_3 a_4 a_5}{a_0^3} \\ \zeta_2 &= \frac{a_1 a_2}{a_5^2}\end{aligned}\tag{5.4}$$

In terms of  $\theta_1 = \zeta_1 \frac{\partial}{\partial \zeta_1}$  and  $\theta_2 = \zeta_2 \frac{\partial}{\partial \zeta_2}$ , the differential operators are:

$$\begin{aligned}\mathcal{D}_1 &= \theta_1^2 (\theta_1 - 2\theta_2) - 3^3 \zeta_1 \prod_{i=1}^3 (\theta_1 + i/3) \\ \mathcal{D}_2 &= \theta_2^2 - \zeta_2 \prod_{i=0}^1 (\theta_1 - 2\theta_2 - i)\end{aligned}\tag{5.5}$$

The system (5.5) can be shown to have 6 linearly independent solutions. We see that in the limit  $\zeta_3 \rightarrow 0$  the system (5.1) reduces to (5.5), while if we also let  $\zeta_2 \rightarrow 0$  the operator  $\mathcal{D}_1$  reduces to the  $\mathcal{D}$  operator in (3.3).

Obviously, one would expect the solutions for these hypergeometric systems to obey the same rule, *i.e.*, taking the limit  $\zeta_2 \rightarrow 0$  in the semi-period expression for  $K3$ , one would get the semi-period expression for  $T^2$ . Similarly, the limit  $\zeta_3 \rightarrow 0$  in the semi-periods for  $\mathbb{P}_{(1,1,2,4,4)}$ [12] will yield  $K3$  semi-periods. To show this, we will analytically continue our expressions to the regions where the desired parameter is small, and then take the zero limit of that parameter.

First, we need to calculate semi-periods. Working in the region of the moduli space where  $p_0$  represents the Fermat part of the defining polynomial, and dropping some numerical factors, one has the following:

–for the torus

$$F_{T^2} = \frac{1}{a_0} \sum_{m=0}^{\infty} \lambda^{(\delta_2+\delta_3)(m+1)} \left( \frac{1}{\zeta_1} \right)^{\frac{m+1}{3}} \frac{(-1)^m}{m!} \Gamma^3 \left( \frac{m+1}{3} \right) ; \quad (5.6)$$

–for the  $K3$

$$\begin{aligned} F_{K3} = & \frac{1}{a_0} \sum_{m=0}^{\infty} \lambda^{(\delta_2+2\delta_3+2\delta_4)(m+1)} \left( \frac{1}{\zeta_1} \right)^{\frac{m+1}{3}} \frac{(-1)^m}{m!} \Gamma^2 \left( \frac{m+1}{3} \right) \\ & \times \left( \frac{1}{\zeta_2} \right)^{\frac{m+1}{6}} \sum_{n=0}^{\infty} \lambda^{3n\delta_2} \left( \frac{1}{\zeta_2} \right)^{\frac{n}{2}} \frac{(-1)^n}{n!} \Gamma^2 \left( \frac{m+3n+1}{6} \right) ; \end{aligned} \quad (5.7)$$

–for the Calabi–Yau

$$\begin{aligned} F_{CY} = & \frac{1}{a_0} \sum_{m=0}^{\infty} \lambda^{(\delta_2+2\delta_3+4\delta_4+4\delta_5)(m+1)} \left( \frac{1}{\zeta_1} \right)^{\frac{m+1}{3}} \frac{(-1)^m}{m!} \Gamma^2 \left( \frac{m+1}{3} \right) \\ & \times \left( \frac{1}{\zeta_2} \right)^{\frac{m+1}{6}} \sum_{n=0}^{\infty} \lambda^{3n(\delta_2+2\delta_3)} \left( \frac{1}{\zeta_2} \right)^{\frac{n}{2}} \frac{(-1)^n}{n!} \Gamma \left( \frac{m+3n+1}{6} \right) \\ & \times \left( \frac{1}{\zeta_3} \right)^{\frac{m+3n+1}{12}} \sum_{p=0}^{\infty} \lambda^{6p\delta_2} \left( \frac{1}{\zeta_3} \right)^{\frac{p}{2}} \frac{(-1)^p}{p!} \Gamma^2 \left( \frac{m+3n+6p+1}{12} \right) . \end{aligned} \quad (5.8)$$

Here  $\lambda = e^{\frac{2\pi i}{d}}$ ,  $d$  being the degree of the polynomial (3, 6 or 12 respectively), and  $\delta_i = 0, 1, \dots, \frac{d}{k_i} - 1$ . This means our contours connect the points  $0, \lambda^{k_i\delta_i} \infty$  in the  $x_i$  plane. Not all different chains lead to different semi-periods; from the above equations one can see the number of linearly independent semi-periods that can be obtained is 3 for the torus, 6 for  $K3$  and 12 for the three-dimensional Calabi–Yau manifold. These then are all the

solutions of the respective hypergeometric equations. To see that the solutions (5.8) are of the type (5.3), one can break up the summations in (5.8) over  $m$ ,  $n$  and  $p$  such that  $m = 3\tilde{m} + i$  for  $i = 0, 1, 2$ ,  $n = 2\tilde{n} + j$  for  $j = 0, 1$ , and  $p = 2\tilde{p} + k$  for  $k = 0, 1$ . Now the summation over  $m$  becomes a summation from 0 to  $\infty$  over  $\tilde{m}$  and a summation from 0 to 2 for  $i$ , and likewise for the others. Now it is clear that the semi-period separates into 12 summations with different leading order behaviours as the  $1/\zeta_i \rightarrow 0$ . These leading orders are those predicted by the solutions of (5.2).

For convenience in doing the analytic continuation we set  $\delta_2 = 0$  and focus on the  $\zeta_3$  dependent part of (5.8). Its Barnes integral representation is as follows [31]:

$$\frac{-1}{2\pi i} \int_0^{i\infty} ds \left( \frac{1}{\zeta_3} \right)^{\frac{m+3n+6s+1}{12}} \Gamma(-s) \Gamma^2 \left( \frac{m+3n+6s+1}{12} \right). \quad (5.9)$$

The poles of  $\Gamma(-s)$  lie along the positive axis. If the contour of integration encloses these poles, then we obtain our starting expression, since for  $s \rightarrow -p$

$$\text{Res } \Gamma(s) = \frac{(-1)^{p+1}}{p!}.$$

The poles of  $\Gamma^2 \left( \frac{m+3n+6s+1}{12} \right)$  lie at  $\frac{m+3n+6s+1}{12} = -k$ , where  $k = 0, 1, 2, 3, \dots$ . If we enclose these poles in the integrating contour (*i.e.*, close the contour to the left) the value of the integral (5.9) will be:

$$-\sum_{k=0}^{\infty} \Gamma \left( \frac{12k+m+3n+1}{6} \right) \left( \frac{1}{\zeta_3} \right)^{-k} \left( \frac{1}{k!} \right)^2 2\psi_1(1+k). \quad (5.10)$$

Here [27]

$$\psi_1(k+1) = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{k} - \gamma,$$

and  $\gamma$  is the Euler-Mascheroni constant. We have also used the fact that

$$\Gamma(-k+\epsilon) = \frac{(-1)^k}{k!} \left( \frac{1}{\epsilon} + \psi_1(1+k) + O(\epsilon) \right)$$

Thus we see that the limit  $\zeta_3 \rightarrow 0$  in (5.8) will give an expression proportional to (5.7).

In a similar manner, one can show that by analytically continuing (5.7) to small  $\zeta_2$ , the limit  $\zeta_2 \rightarrow 0$  is proportional to (5.6).

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